**Q2 (a)** **If z be a homogeneous function of degree n, show that**

$$x\frac{\partial^2 z}{\partial x \cdot \partial y} + y\frac{\partial^2 z}{\partial y^2} = (n-1)\frac{\partial z}{\partial y}$$

**Answer**

Soln: Q. 2. a.

By Euler's Theorem

$$x \cdot \frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz \quad \text{(i)}$$

Differentiating (i), partially w.r.t. 'x', we get

$$\frac{\partial z}{\partial x} + x \cdot \frac{\partial^2 z}{\partial x^2} + y \cdot \frac{\partial^2 z}{\partial x \cdot \partial y} = n\frac{\partial z}{\partial x}$$

$$\text{or} \quad x \cdot \frac{\partial^2 z}{\partial x^2} + y \cdot \frac{\partial^2 z}{\partial x \cdot \partial y} = (n-1)\frac{\partial z}{\partial x} \quad \text{(ii)}$$

Differentiating (i), partially w.r.t. 'y', we have

$$x \cdot \frac{\partial^2 z}{\partial y \cdot \partial x} + \frac{\partial z}{\partial y} + y \cdot \frac{\partial^2 z}{\partial y^2} = n\frac{\partial z}{\partial y}$$

$$\text{or} \quad x \cdot \frac{\partial^2 z}{\partial y \cdot \partial x} + y\frac{\partial^2 z}{\partial y^2} = (n-1)\frac{\partial z}{\partial y}$$

Hence proved

**Q2 (b)** **If** $z = f(x, y)$ **, where** $x = e^u \cos v$ **and** $y = e^u \sin v$ **, then show that**

$$y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v} = e^{2u}\frac{\partial z}{\partial y}$$

**Answer**

2. b. Soln:

$$x = e^u \cos v, \quad y = e^u \sin v$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$= \frac{\partial z}{\partial x} e^u \cos v + \frac{\partial z}{\partial y} e^u \sin v$$

$$= x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$$

$$y\frac{\partial z}{\partial u} = xy\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} \quad \text{--- (i)}$$

and
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$= \frac{\partial z}{\partial x}(-e^u \sin v) + \frac{\partial z}{\partial y}(e^u \cos v)$$

$$= -y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y}$$

$$x\frac{\partial z}{\partial v} = -xy\frac{\partial z}{\partial x} + x^2\frac{\partial z}{\partial y} \quad \text{--- (ii)}$$

On adding (i) and (ii), we get

$$y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v} = (x^2+y^2)\frac{\partial z}{\partial y}$$

$$= (e^{2u}\cos^2 v + e^{2u}\sin^2 v)\frac{\partial z}{\partial y}$$

$$= e^{2u}(\cos^2 v + \sin^2 v)\cdot\frac{\partial z}{\partial y}$$

$$= e^{2u}\frac{\partial z}{\partial y}$$

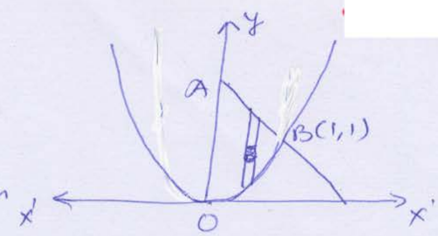
Hence proved

**Q3 (a) Change the order of integration and then evaluate** $I = \int_0^1 \int_{x^2}^{2-x} xy\,dx\,dy$

**Answer**

Q3.a. Soln.

$I = \int_0^1 \int_{x^2}^{2-x} xy \, dx \, dy$

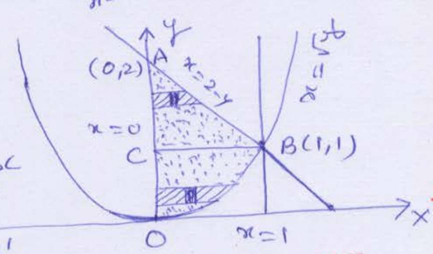



The region of integration is shown by shaded portion in the figure bounded by parabola $y = x^2$ and the line $y = 2-x$

The point of intersection of the parabola $y = x^2$ and the line $y = 2-x$ is $(1,1)$.

In the fig. we have taken a strip parallel to $y$-axis and the order of integration is $\int_0^1 x\, dx \int_{x^2}^{2-x} y \, dx$

In the second figure we have taken a strip parallel to $x$-axis in the area OBC and second strip in the area ABC.




The limits of $x$ in the area OBC are 0 and $\sqrt{y}$ and the limit of $x$ in the area ABC are 0 and $2-y$.

So, the given integral is

$$= \int_0^1 y \, dy \int_0^{\sqrt{y}} x \, dx + \int_1^2 y \, dy \int_0^{2-y} x \, dx$$

$$= \int_0^1 y \, dy \left[\frac{x^2}{2}\right]_0^{\sqrt{y}} + \int_1^2 y \, dy \left[\frac{x^2}{2}\right]_0^{2-y}$$

$$= \frac{1}{2}\int_0^1 y^2 \, dy + \frac{1}{2}\int_1^2 y(2-y)^2 dy$$

$$= \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1 + \frac{1}{2}\int_1^2 (4y - 4y^2 + y^3) \, dy$$

$$= \frac{1}{6} + \frac{1}{2}\left[2y - \frac{4}{3}y^3 + \frac{y^4}{4}\right]_1^2$$

$$= \frac{1}{6} + \frac{1}{2}\left[8 - \frac{32}{3} + 4 - 2 + \frac{4}{3} - \frac{1}{4}\right]$$

$$= \frac{1}{6} + \frac{1}{2}\left[\frac{96 - 128 + 48 - 24 + 16 - 3}{12}\right]$$

$$= \frac{1}{6} + \frac{5}{24} = \frac{9}{24} = \frac{3}{8} \text{ Ans.}$$

**Q3 (b) Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 3$ and z = 0**

**Answer**

Q.3 b. Soln.

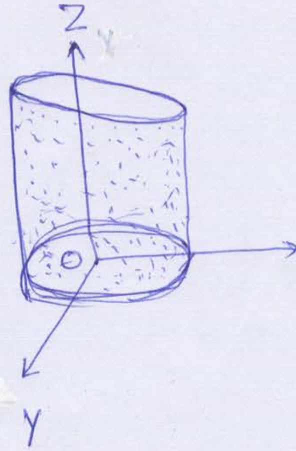
$$x^2+y^2=4 \quad \text{(i)}$$
$$y+z=3 \quad \text{(ii)}$$
$$z=0 \quad \text{(iii)}$$

(i) $z$ varies from $0$ to $3-y$

(ii) $y$ varies from $-\sqrt{4-x^2}$ to $\sqrt{4-x^2}$

(iii) $x$ varies from $-2$ to $+2$

The required volume




$$= \int_{-2}^{+2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{3-y} dx\,dy\,dz$$

$$= \int_{-2}^{+2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \Big[z\Big]_0^{3-y} dx.dy$$

$$= \int_{-2}^{+2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x}} \Big[3-y\Big] dx\,dy$$

$$= \int_{-2}^{+2} \left[3y-\frac{y^2}{2}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{+2}\left[3\sqrt{4-x^2} - \frac{4-x^2}{2} + 3\sqrt{4-x^2} + \frac{4-x^2}{2}\right] dx$$

$$= 6\int_{-2}^{+2}\sqrt{4-x^2}\,dx = 6\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_{-2}^{+2}$$

$$= 6\left[2\sin^{-1}\frac{2}{2} - 2\sin^{-1}\frac{-2}{2}\right] = 12\left[\frac{\pi}{2}+\frac{\pi}{2}\right] = 12\pi \quad \text{Ans.}$$

marks

Total = 8 marks.

**Q4 (a) Find the eigenvalues and eigenvectors of the matrix** $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

**Answer**

Soln: $|A - \lambda I| = 0$

$$\begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix} = 0 \quad \text{i.e., } \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

let $\lambda = 1$, $1 - 6 + 11 - 6 = 0$

By synthetic division

$$\begin{array}{r|rrrr} 1 & 1 & -6 & +11 & -6 \\ & & -1 & -5 & 6 \\ \hline & 1 & -5 & 6 & 0 \end{array}$$

$\lambda^2 - 5\lambda + 6 = 0$

P.T.O.

$(\lambda - 1)(\lambda^2 - 5\lambda + 6) = 0$

$(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0 \Rightarrow \lambda = 1, 2, 3$

To find eigenvectors for the corresponding eigenvalues we will consider the matrix equation

$(A - \lambda I)X = 0$

$$\begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{——— (1)}$$

Eigenvector corresponding to eigenvalue $\lambda = 1$ by putting $\lambda = 1$, the matrix equation (1) will become

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} -z = 0 \\ x + y + z = 0, \text{ Let } x = k \\ k + y + 0 = 0 \quad y = -k \end{matrix}$$

Eigenvector $X_1$ is $\begin{bmatrix} k \\ -k \\ 0 \end{bmatrix}$ or $k\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

Eigenvector corresponding to eigenvalue $\lambda = 2$ on putting $\lambda = 2$ in eq. (1), it will become

$$\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} -x + 0y - z = 0 \\ 2x + 2y + z = 0 \end{matrix}$$

$$\frac{x}{0+2} = \frac{y}{-2+1} = \frac{z}{-2-0} \quad \text{or} \quad \frac{x}{2} = \frac{y}{-1} = \frac{z}{-2}$$

Eigenvector $X_1 = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$

Eigenvector corresponding to eigenvalue $\lambda = 3$ will become on putting $\lambda = 3$ in (1), the equation

$$\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} -2x + 0 \cdot y - z = 0 \\ x - y + z = 0 \end{matrix} \Rightarrow \frac{x}{0-1} = \frac{y}{-1+2} = \frac{z}{2-0}$$

or $\frac{x}{1} = \frac{y}{-1} = \frac{z}{-2}$

Eigenvector $X_3 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$ Ans.

**Q4 (b) Determine the rank of the following matrices**

(i) $\begin{bmatrix} 1 & 4 & 5 \\ 2 & 6 & 8 \\ 3 & 7 & 22 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

**Answer**

Q. 4. b. Soln

(i) $\begin{bmatrix} 1 & 4 & 5 \\ 2 & 6 & 8 \\ 3 & 7 & 22 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 \\ 0 & -2 & -2 \\ 0 & -5 & 7 \end{bmatrix} \begin{matrix} R_2-2R_1 \\ R_3-3R_1 \end{matrix}$

$\sim \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 1 \\ 0 & -5 & 7 \end{bmatrix} -\frac{1}{2}R_2$

$\sim \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 12 \end{bmatrix} R_3+5R_2$

Ans: 3

(ii) $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

$R_1 \leftrightarrow R_2$

$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

$R_2-2R_1,\ R_3-3R_1,\ R_4-6R_1$

$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$

$R_3-\frac{4}{5}R_2,\ R_4-\frac{9}{5}R_2$

$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 33/5 & 22/5 \\ 0 & 0 & 33/5 & 22/5 \end{bmatrix}$

$R_4-R_3$

$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 33/5 & 22/5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Rank = Number of Non-zero rows
= 3 Ans

**Q5 (a) Find the approximate value of the root of the equation $x^3+x-1=0$ near $x=1$, using the method of Regula-Falsi two times.**

**Answer**

Soln:

$f(x) = x^3 + x - 1 = 0$

$f(1) = 1+1-1 = +1$

$f(.5) = (.5)^3 + (.5) - 1 = -.375$

The root lies between .5 and 1.

Let $x_1 = 0.5$ and $x_2 = 1$

$$x = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} \quad \text{or} \quad x = \frac{0.5 f(1) - 1 \cdot f(0.5)}{f(1) - f(0.5)}$$

$$= \frac{0.5(1) - 1(-0.375)}{1 + 0.375} = 0.64$$

Now, $f(0.64) = -0.0979$ and $f(1) = 1$

$\therefore$ Root lies between .64 and 1

$x_1 = 0.64$ and $x_2 = 1$

$$x = \frac{0.64 f(1) - 1 f(0.64)}{f(1) - f(0.64)} = \frac{0.672 + 0.0245}{1 + 0.0245} = 0.6822$$

Ans

**Q5 (b) Express the following system of equations in matrix form and solve them by the elimination method due to Gauss:**

$$\begin{aligned}
2x_1 + x_2 + 2x_3 + x_4 &= 6 \\
6x_1 - 6x_2 + 6x_3 + 12x_4 &= 36 \\
4x_1 + 3x_2 + 3x_3 - 3x_4 &= -1 \\
2x_1 + 2x_2 - x_3 + x_4 &= 10
\end{aligned}$$

**Answer**

The equation are expressed in matrix as,

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 6 & -6 & 6 & 12 \\ 4 & 3 & 3 & -3 \\ 2 & 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 36 \\ -1 \\ 10 \end{bmatrix} \text{ or } \begin{bmatrix} 2 & 1 & 2 & 1 \\ 0 & -9 & 0 & 9 \\ 0 & 1 & -1 & -5 \\ 0 & 1 & -3 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 18 \\ -13 \\ 4 \end{bmatrix} \begin{matrix} R_2-3R_1 \\ R_3-2R_1 \\ R_4-R_1 \end{matrix}$$

$$\text{or } \begin{bmatrix} 2 & 1 & 2 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & -5 \\ 0 & 1 & -3 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ -13 \\ 4 \end{bmatrix} R_2 \to \frac{R_2}{-9}$$

$$\text{or } \begin{bmatrix} 2 & 1 & 2 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & -3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ -11 \\ 6 \end{bmatrix} \begin{matrix} R_3-R_2 \\ R_4-R_2 \end{matrix}$$

$$\text{or } \begin{bmatrix} 2 & 1 & 2 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & 0 & 13 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ -11 \\ 39 \end{bmatrix} R_4-3R_3$$

$$2x_1+x_2+2x_3+x_4=6 \quad \text{(i)}$$
$$x_2-x_4=-2 \quad \text{(ii)}$$
$$-x_3-4x_4=-11 \quad \text{(iii)}$$

$13x_4=39$ or $x_4=3$

Putting the value of $x_4$ in (iii) we get

$-x_3-12=-11$ or $x_3=-1$

Putting the value of $x_4$ in (ii) we get

$x_2-3=-2$ or $x_2=1$

Substituting the value of $x_4, x_3, x_2$ in (i) we get

$2x_1+1-2+3=6$ or $2x_1=4$ or $x_1=2$

$\therefore\ x_1=2,\ x_2=1,\ x_3=-1,\ x_4=3$ Ans

**Q6 (a) Solve the differential equation** $(D-2)^2 y = 8\left(e^{2x}+\sin 2x + x^2\right)$ **where** $D \equiv \dfrac{d}{dx}$

**Answer**

Soln:

To find complementary function C.F.

It's A.E. is $(D-2)^2=0$, $\therefore D=2,2$

Thus $C.F. = (C_1+C_2x)\,e^{2x}$

To find Particular Integral P.I.

$$P.I. = 8\left[\frac{1}{(D-2)^2}e^{2x} + \frac{1}{(D-2)^2}\sin 2x + \frac{1}{(D-2)}x^2\right]$$

Now, $\frac{1}{(D-2)^2}e^{2x} = x^2 \cdot \frac{1}{2(1)} e^{2x}$ $\left[\because \text{by putting } D=2, (D-2)^2=0, 2(D-2)=0\right.$

$$= \frac{x^2 e^{2x}}{2}$$

$$\frac{1}{(D-2)^2}\sin 2x = \frac{1}{D^2-4D+4}\sin 2x = \frac{1}{(-2)^2-4D+4}\sin 2x$$

$$= -\frac{1}{4}\int \sin 2x\,dx = -\frac{1}{4}\left(\frac{-\cos 2x}{2}\right)$$

$$= \frac{1}{8}\cos 2x$$

and $\frac{1}{(D-2)^2}x^2 = \frac{1}{4}\left(1-\frac{D}{2}\right)^{-2}\cdot x^2$

$$= \frac{1}{4}\left[1+(-2)\left(\frac{D}{2}\right)+\frac{(-2)(-3)}{2!}\left(-\frac{D}{2}\right)^2+\cdots\right]x^2$$

$$= \frac{1}{4}\left[1+D+\frac{3D^2}{4}+\cdots\right]x^2$$

$$= \frac{1}{4}\left[x^2+2x+\frac{3}{2}\right]$$

Thus, $P.I. = 4x^2e^{2x} + \cos 2x + 2x^2 + 4x + 3$

Hence the complete solution,

$$y = (C_1+C_2x)e^{2x} + 4x^2e^{2x} + \cos 2x + 2x^2 + 4x + 3$$

Ans.

**Q6 (b) Solve the equation,** $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \log x$

**Answer**

b) $y = ($

Total: 8 mks Ans:

Q.7.b. Soln:

This is a Cauchy's homogeneous linear, Put $x = e^t$, i.e. $t = \log x$, so that,

$$x\frac{dy}{dx} = Dy,\ x^2\frac{d^2y}{dx^2} = D(D-1)y \quad \text{where } D = \frac{d}{dt}$$

4 mks.

Then the given equation becomes

$$[D(D-1) - D + 1]y = t \text{ or } (D-1)^2 y = t \quad \text{--- (1)}$$

which is a linear equation with constant coefficients,
Its A.E. is $(D-1)^2 = 0$, whence $D = 1, 1$

$\therefore$ C.F. $= (c_1 + c_2 t)e^t$ and

$$P.I. = \frac{1}{(D-1)^2}t = (1-D)^{-2}t$$
$$= (1 + 2D + 3D^2 + \cdots)$$
$$= t + 2$$

4 mks

Hence the solution of (1) is $y = (c_1 + c_2 t)e^t + t + 2$

or putting $t = \log x$ and $e^t = x$, we get

$y = (c_1 + c_2 \log x)x + \log x + 2$ as the reqd. solution of (1)

**Q7 (a) Obtain the series solution of** $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + xy = 0$

**Answer**

Soln:

Substituting $y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \cdots$ — (i)

$$\frac{dy}{dx} = m a_0 x^{m-1} + (m+1) a_1 x^m + (m+2) a_2 x^{m+1} + \cdots$$

and $$\frac{d^2y}{dx^2} = m(m-1) a_0 x^{m-2} + (m+1) m a_1 x^{m-1} + (m+2)(m+1) a_2 x^m + \cdots$$

in the given equation, we obtain

$$x\left[m(m-1)a_0 x^{m-2} + (m+1) m a_1 x^{m-1} + (m+2)(m+1) a_2 x^m + \cdots\right]$$
$$+ \left[m a_0 x^{m-1} + (m+1) a_1 x^m + (m+2) a_2 x^{m+1} + \cdots\right]$$
$$+ x\left[a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \cdots\right] = 0$$

The lowest power of $x$ is $x^{m-1}$. Its coefficient equated to zero gives $a_0[m(m-1) + m] = 0$

i.e, $m^2 = 0$ as $a_0 \neq 0$, $\therefore m = 0, 0$

The coefficient of $x^m, x^{m+1}, \ldots$, equated to zero give

$a_1[(m+1)m + m + 1] = 0$, i.e $a_1 = 0$

$a_2(m+2)^2 + a_0 = 0$, $a_3(m+3)^2 + a_1 = 0$, $a_4(m+4)^2 + a_2 = 0$

and so on.

Clearly, $a_3 = a_5 = a_7 \cdots = 0$

Also, $a_2 = -\dfrac{a_0}{(m+2)^2}$

$$a_4 = -\frac{a_2}{(m+4)^2} = \frac{a_0}{(m+2)^2(m+4)^2} \text{ etc}$$

$$\therefore y = a_0 x^m\left[1 - \frac{x^2}{(m+2)^2} + \frac{x^4}{(m+2)^2(m+4)^2} - \frac{x^6}{(m+2)^2(m+4)^2(m+6)^2} + \cdots\right] \quad \text{— (ii)}$$

Putting $m = 0$, the first solution is

$$y_1 = a_0\left[1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots\right] \quad \text{— (iii)}$$

This gives only one solution instead of two.

To get the second, differentiate (ii) partially w.r.t. $m$

$$\frac{\partial y}{\partial m} = y \log x + a_0 x^m \left\{\frac{x^2}{(m+2)^2}\frac{2}{m+2} - \frac{x^4}{(m+2)^2(m+4)^2}\left[\frac{2}{m+2} + \frac{2}{m+4}\right] + \cdots\right\}$$

P.T.O.

∴ The second solution is $y_2 = \left(\frac{\partial y}{\partial m}\right)_{m=0}$

$$= y_1 \log x + a_0 \left\{ \frac{1}{2^2}x^2 - \frac{1}{2^2.4^2}\left(1+\frac{1}{2}\right)x^4 + \frac{1}{2^2.4^2.6^2}\left(1+\frac{1}{2}+\frac{1}{3}\right)x^6 - \dots \right\} \quad \text{—— (iv)}$$

Hence the complete solution is

$$y = c_1 y_1 + c_2 y_2$$

i.e. $$y = (C_1 + C_2 \log x)\left[1 - \frac{1}{2^2}x^2 + \frac{1}{2^2.4^2}x^4 - \frac{1}{2^2.4^2.6^2}x^6 + \dots\right]$$
$$+ C_2\left\{\frac{1}{2^2}x^2 - \frac{1}{2^2.4^2}\left(1+\frac{1}{2}\right)x^4 + \frac{1}{2^2.4^2.6^2}\left(1+\frac{1}{2}+\frac{1}{3}\right)x^6 - \dots\right\}$$

where $C_1 = a_0 c_1$, $C_2 = a_0 c_2$ Ans

**Q7 (b) State and prove orthogonality of Legendre polynomials.**
**Answer**

Soln:

We shall prove that, $\int_{-1}^{1} P_m(x) P_n(x)\,dx = \begin{cases} 0, & m \neq n \\ \frac{2}{2n+1}, & m = n \end{cases}$

We know that the solution of

$(1-x^2)u'' - 2xu' + m(m+1)u = 0$ —— (1)

and $(1-x^2)v'' - 2xv' + n(n+1)v = 0$ —— (2)

are $P_m(x)$ and $P_n(x)$ respectively.

Multiplying (1) by $v$ and (2) by $u$ and subtracting, we get

$(1-x^2)(u''v - uv'') - 2x(u'v - uv') + [m(m+1) - n(n+1)]uv = 0$

$\frac{d}{dx}[(1-x^2)(u'v - uv')] + (m-n)(m+n+1)uv = 0$

Now integrating from $-1$ to $1$, we get

$(m-n)(m+n+1)\int_{-1}^{1} uv\,dx = \left|(1-x^2)(uv' - u'v)\right|_{-1}^{1} = 0$

Hence $\int_{-1}^{1} P_m(x) P_n(x)\,dx = 0. \quad (m \neq n)$

This is known as the orthogonality property of Legendre Polynomials.

Where $m = n$, we have from Rodrigues's formula

$(n!\,2^n)^2 \int_{-1}^{1} P_n^2(x)\,dx = \int_{-1}^{1} D^n(x^2-1)^n \cdot D^n(x^2-1)^n\,dx$ (Integrating by parts)

$= \left| D^n(x^2-1)^n \, D^{n-1}(x^2-1)^n \right|_{-1}^{1} - \int_{-1}^{1} D^{n+1}(x^2-1)^n \, D^{n-1}(x^2-1)^n\,dx$

Since $D^{n-1}(x^2-1)^n$ has $x^2-1$ as a factor, the first term on the right vanishes for $x = \pm 1$.

**Q8 (a) Obtain Fourier series for the function**

$$f(x) = \begin{cases} \pi x, & 0 \le x \le 1 \\ \pi(2-x), & 1 \le x \le 2 \end{cases}$$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\dots\dots\infty = \frac{\pi^2}{8}$

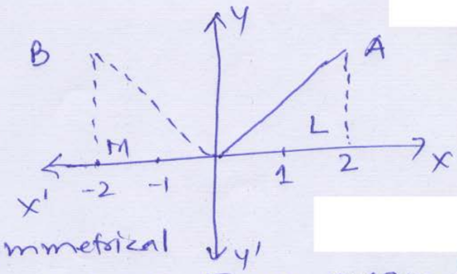
**Answer** Page numbers 406 from text book

**Q8 (b) Express $f(x) = x$ as a half-range cosine series in $0 < x < 2$**

**Answer**

Soln.

The graph of $f(x) = x$ in $(0,2)$ is the line OA. Let us extend the function $f(x)$ in the interval $(-2,0)$ shown by the line OB' so that the new function is symmetrical about the x-axis and, therefore, represents an even function in $(-2,2)$ in fig.




Hence the fourier series for $f(x)$ over the full period $(-2,2)$ will contain only cosine terms given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}$$

where $a_0 = \frac{2}{2}\int_0^2 f(x)\,dx = \int_0^2 x\,dx = 2$

and $a_n = \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx$

$$= \int_0^2 x\cos\frac{n\pi x}{2}\,dx$$

$$= \left| \frac{2x}{n\pi}\sin\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2} \right|_0^2$$

$$= \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right]$$

P.T.O.

Thus $a_1 = -8/\pi^2$, $a_2 = 0$, $a_3 = -8/3^2\pi^2$, $a_4 = 0$,

$a_5 = -8/5^2\pi^2$ etc.

Hence the desired fourier series for $f(x)$ over the half-range $(0,2)$ is 4mks.

$$f(x) = 1 - \frac{8}{\pi^2}\left[\frac{\cos \pi x/2}{1^2} + \frac{\cos 3\pi x/2}{3^2} + \frac{\cos 5\pi x/2}{5^2} + \cdots\right].$$

Ans.

**Q9 (a) State and prove Convolution theorem for Fourier transforms.**

**Answer** Page numbers 777 from text book

**Q9 (b) Solve by z-transform** $y_{k+1} + \frac{1}{4} yk = \left(\frac{1}{4}\right)^k \left[k \geq 0, y(0) = 0\right]$

**Answer**

Soln: $y_{k+1} + \frac{1}{4} y_k = \left(\frac{1}{4}\right)^k \quad \text{--- (i)}$

Taking Z-transform of both sides of (i), we get

$$Z\left[y_{k+1} + \frac{1}{4} y_k\right] = Z\left[\left(\frac{1}{4}\right)^k\right]$$

$$Z\left[y_{k+1}\right] + Z\left[\frac{1}{4} y_k\right] = Z\left[\left(\frac{1}{4}\right)^k\right]$$

$$z y(z) - z y(0) + \frac{1}{4} y(z) = \frac{1}{1 - \frac{1}{4} z^{-1}} \qquad \left[\because |z| > \frac{1}{4}\right]$$

$$z y(z) - 0 + \frac{1}{4} y(z) = \frac{1}{1 - \frac{1}{4} z^{-1}}$$

$$\left(z + \frac{1}{4}\right) y(z) = \frac{1}{1 - \frac{1}{4} z^{-1}}$$

$$y(z) = \frac{1}{z + \frac{1}{4}} \times \frac{1}{1 - \frac{1}{4} z^{-1}} = \frac{z^{-1}}{1 + \frac{1}{4} z^{-1}} \times \frac{1}{1 - \frac{1}{4} z^{-1}}$$

$$= \frac{-2}{1 + \frac{1}{4} z^{-1}} + \frac{2}{1 - \frac{1}{4} z^{-1}}$$

$$y_k = z^{-1}\left[\frac{-2}{1 + \frac{1}{4} z^{-1}}\right] + z^{-1}\left[\frac{2}{1 - \frac{1}{4} z^{-1}}\right]$$

$$= z^{-1}\left[-2\left(1 + \frac{1}{4} z^{-1}\right)^{-1}\right] + z^{-1}\left[2\left(1 - \frac{1}{4} z^{-1}\right)^{-1}\right]$$

$$= -2\left(-\frac{1}{4}\right)^k + 2\left(\frac{1}{4}\right)^k \qquad \text{Ans}$$

## Text Book

**Higher Engineering Mathematics, Dr. B S Grewal, 41st Edition 2012, Khanna Publishers, Delhi.**